Topological Structures on Fuzzy Multisets

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Abstract: In this paper the concept of fuzzy topological space extended into fuzzy multiset topological space, closed fuzzy multiset, open fuzzy multisets, interior fuzzy multisets, closure of fuzzy multisets and their various properties discuss. Further, we extended the continuous functions to fuzzy multiset continuous function and also discussed subspace of fuzzy multiset topological space.

Keywords: Fuzzy multiset, fuzzy multiset topological space, open fuzzy multiset, closed fuzzy multiset, interior fuzzy multiset, closure fuzzy multiset, fuzzy multiset continuous function and fuzzy multiset subspace.

1. Introduction

The concept fuzzy set theory is introduced by L. A. Zadeh [1]. Chang [2] introduced fuzzy topological space in the year of 1968. Many author has been discuss the theory and application of fuzzy topological spaces. Blizard [3] traced multisets back to the very origin of numbers, arguing that in ancient times, the number n was often represented by a collection of n strokes, tally marks, or units. The idea of fuzzy multiset was introduced by Yager [4] as fuzzy bags. In the interest of brevity, we shall consider our attention in the basic concepts such as an open fuzzy multiset (OFMS), closed fuzzy multiset (CFMS), interior, closure and continuity of fuzzy multisets. Our notation and terminology for fuzzy multiset follows [4] and [5].

2. Preliminaries

2.1 Definition: If \( X \) is a collection of objects denoted by \( x \) then a fuzzy set \( A \) in \( X \) is a set of ordered pairs

\[ A = \{ (x, \mu_A(x)) : x \in X \} \]

Where \( \mu_A(x) \) is called the membership function.

2.2 Definition: Let X be a non-empty set. A Fuzzy multiset (FMS) A drawn from X is characterized by a function, ‘count membership’ of A denoted by \( CM_A(x) \) such that \( CM_A: X \rightarrow Q \) where Q is the set of all crisp multisets drawn from the unit interval[0,1]. Then for any \( x \in X \), the value \( CM_A(x) \) is crisp multiset drawn from [0,1]. For each \( x \in X \), the membership sequence is defined as decreasing ordered sequence of elements in \( CM_A(x) \). It is denoted by \( \mu^1_A(x), \mu^2_A(x), \mu^3_A(x), ..., \mu^n_A(x) \) where

\[ \mu^1_A(x) \geq \mu^2_A(x) \geq \mu^3_A(x) \geq ..., \geq \mu^n_A(x) \]

2.3 Definition: Length of an element x in an FMS A is defined as the cardinality of \( CM_A(x) \) and it is denoted by

\[ L(x:A) = |CM_A(x)| \]

2.4 Definition: If A and B are FMS’s drawn from X then

\[ L(x:A,B) = \text{Max}\{ L(x:A), L(x:B) \} \]

we can use notation L(x) for L(x:A,B).

2.5 Example: Consider X={x,y,z,w}

\[ A = \{(0.3,0.2)/x,(1,0.5,0.5)/y,(0.5,0.4,0.3,0.2)/z\} \]

\[ B = \{(0.2,0.1)/x,(1,0.3,0.2)/y,(0.2,0.1)/w\} \]

Then

\[ L(x:A) = 2, L(y:A) = 3, L(z:A) = 4 \]

\[ L(x:B) = 2, L(y:B) = 3, L(w:B) = 2 \]

\[ L(x:A,B) = \text{Max}\{ L(x:A), L(x:B) \} = 2 \]

\[ L(y:A,B) = 3, L(z:A,B) = 4, L(w:A,B) = 2 \]

3 Basic Operations On Fuzzy Multisets (FMS)

3.1 Definition: For any two FMSs A and B drawn from a decreasing X, the following operations and relations will hold. Let
A = \{(\mu_A^1(x), \mu_A^2(x), \mu_A^3(x), ..., \mu_A^P(x))/x: x \in X\} 
\text{ and } 
B = \{(\mu_B^1(x), \mu_B^2(x), \mu_B^3(x), ..., \mu_B^P(x))/x: x \in X\}

1. **Inclusion**
   \[ A \subseteq B \iff \mu_A^i(x) \leq \mu_B^i(x) \text{ for } i = 1,2,...,L(x), x \in X \]
   \[ A = B \iff A \subseteq B \text{ and } B \subseteq A \]

2. **Complement**
   \[ A^c = \{1 - \mu_A^1(x), 1 - \mu_A^2(x), 1 - \mu_A^3(x), ..., 1 - \mu_A^P(x) / x: x \in X\} \]

3. **Union** \( A \cup B \)
   In \( A \cup B \) the membership obtained as follows,
   \[ \mu_{A\cup B}^i(x) = \mu_A^i(x) \land \mu_B^i(x) \text{ for } i = 1,2,...,L(x), x \in X \]

4. **Intersection** \( A \cap B \)
   In \( A \cap B \) the membership obtained as follows,
   \[ \mu_{A\cap B}^i(x) = \mu_A^i(x) \lor \mu_B^i(x) \text{ for } i = 1,2,...,L(x), x \in X \]

Here \( \land \) and \( \lor \) denotes maximum and minimum real numbers respectively.

3.1 **Theorem:** For any three FMS’s \( A, B, C \)

   1. **Commutative Law**
      \[ A \cup B = B \cup A \]
      \[ A \cap B = B \cap A \]
   2. **Idempotent Law**
      \[ A \cup A = A \]
      \[ A \cap A = A \]
   3. **De Morgan’s Laws**
      \[ (A \cup B)^c = (A^c \cap B^c) \]
      \[ (A \cap B)^c = (A^c \cup B^c) \]
   4. **Associative Law**
      \[ A \cup (B \cup C) = (A \cup B) \cup C \]
      \[ A \cap (B \cap C) = (A \cap B) \cap C \]
   5. **Distributive Law**
      \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

3.2 **Definition:** Let \( X \) and \( Y \) be two non-empty sets and \( f: X \rightarrow Y \) be a mapping. Then

1. The image of FMS \( A \in FM(X) \) under mapping \( f \) is denoted by \( f(A) \) or \( f[A] \), where
   \[ CM_f(A)(y) = \begin{cases} \mu_A(x) & f(x) = y \\ 0 & \text{otherwise} \end{cases} \]
2. The inverse image of FMS \( B \in FM(Y) \) under the mapping \( f \) is denoted by \( f^{-1}(B) \) or \( f^{-1}[B] \), where \( CM_{f^{-1}}(B)(x) = CM_B(f(x)) \).

4 **Fuzzy Multiset Topological Spaces**

In this section we introduce the concept of fuzzy multiset topology (FMT). Here we extend the concept of fuzzy topological space introduced by C.L. Chang [2] to the case of fuzzy multiset topology. For this we first introduced \( \bar{0} \) and \( \bar{1} \) in a non-empty set \( X \) as follows:

4.1 **Definition:** Let
   \[ \bar{0} = \{(0,0,0,...,0)/x:x \in X\} \text{ and } \bar{1} = \{(1,1,1,...,1)/x:x \in X\} \]

4.2 **Definition:** A fuzzy multiset topology (FMT) on \( X \) is a family \( \Gamma \) of fuzzy multisets (FMS) such that

1. \( \bar{0}, \bar{1} \in \Gamma \)
2. \( F_1 \cap F_2 \in \Gamma \) for any \( F_1, F_2 \in \Gamma \)
3. \( \cup F_i \in \Gamma \) for any arbitrary family \( \{F_i: i \in I\} \) in \( \Gamma \)

then the pair \( (X, \Gamma) \) is called Fuzzy Multiset Topological Space (FMTS) and any FMS in \( \Gamma \) is know as and open fuzzy multiset (OFMS) in \( X \).

4.1 **Remark:** The complement of an OFMS is called closed fuzzy multiset (CFMS).

4.1 **Example:** Let \( X = \{1,2\} \) and defined the FMSs in \( X \) as follows,

For \( n \in \mathbb{N} \)

\[ F_n = \{(1,1/2,1/3,...,1/n)/1, (1/2,1/3,1/4,...,1/(n + 1))/2\} \]

Let \( \Gamma = \{\bar{0}, \bar{1}\} \cup F_n \)

Then \( (X, \Gamma) \) forms an FMT.
4.3 Definition: A FMS-base for a fuzzy multiset
topological space\((X,\Gamma)\) is a sub collection \(B\) of \(\Gamma\)
such that each member \(A\) of \(\Gamma\) can be written as
\(A = \bigvee_{j\in A} A_j\) where each \(A_j \in B\).

4.4 Definition: Let \((X,\Gamma)\) be a FMT and \(A\) be a FMS
in \(X\). Then the closure of \(A\) is denoted by \(cl(A)\) is
defined as \(cl(A) = \bigcap \{M : M is closed in X and A \subseteq M\}\).

4.5 Definition: Let \((X,\Gamma)\) be a FMT and \(A\) be a FMS
in \(X\). Then the interior of \(A\) is denoted by \(int(A)\) is
defined as \(int(A) = \bigcup \{M : M is open in X and M \subseteq A\}\).

4.6 Definition: Let \((X,\Gamma_1)\) and \((X,\Gamma_2)\) be two
FMT’s on \(X\). Then \(\Gamma_1\) is coarser(weaker) than \(\Gamma_2\) if
\(A \in \Gamma_2\) for each \(A \in \Gamma_1\). It is denoted by \(\Gamma_1 \subseteq \Gamma_2\).

4.1 Proposition: Let \(\{\Gamma_i : i \in I\}\) be a family of FMTs
on \(X\). Then \(\bigcap_{i \in I} \Gamma_i\) is a FMT on \(X\) and it is
coarsest FMT on \(X\) containing all \(\Gamma_i\)’s.

Proof follows above two definitions 4.5 and 4.6.

4.2 Proposition: Let \((X,\Gamma)\) be an FMT and \(A\) be an
FMS in \(X\). Then \(cl(A)\) is a CFMS.

Proof: \(cl(A) = \bigcap \{M : M is CFMS in X and A \subseteq M\}\)
\(\Rightarrow (cl(A))^c\) is union of all open set and hence open.
\(\Rightarrow cl(A)\) is CFMS.

4.3 Proposition: Let \((X,\Gamma)\) be an FMT and \(A\) be an
FMS in \(X\). Then \(int(A)\) is an OFMS.

Proof follows by definition 4.5.

4.2 Remark: From above two propositions 4.2, 4.3
and the definition 4.4, 4.5 it is clear that \(cl(A)\) is the
smallest CFMS containing \(A\) and \(int(A)\) is the
largest OFMS contained \(A\).

4.4 Proposition: Let \((X,\Gamma)\) be an FMT and \(A\) be
FMS. Then \(cl(A^c) = (int(A))^c\).

Proof: Let \(A = \{(\mu_{A_i}^1(x), \mu_{A_i}^2(x), \ldots, \mu_{A_i}^p(x))/x : x \in X\}\)
and
\(A_i = \{(\mu_{A_i}^1(x), \mu_{A_i}^2(x), \ldots, \mu_{A_i}^p(x))/x : x \in X, i \in I\}\)
Be the family of OFMSs which contained \(A\). Then
\(\mu_{A_i}^j(x) \leq \mu_{A}^j(x) ; i \in I; j = 1, 2, 3, \ldots, p\)
And \(int(A) = \{(\vee \mu_{A_i}^j(x), \vee \mu_{A_i}^j(x), \ldots, \vee \mu_{A_i}^p(x))/x : x \in X, i \in I\}\)
\(\Rightarrow (int(A))^c = \{(1 - \vee \mu_{A_i}^j(x), 1 - \vee \mu_{A_i}^j(x), \ldots, 1 - \vee \mu_{A_i}^p(x))/x : x \in X, i \in I\}\)

(1)

Now \(A^c = \{(1 - \mu_{A_i}^j(x), 1 - \mu_{A_i}^2(x), \ldots, 1 - \mu_{A_i}^p(x))/x : x \in X\}\) From (1) it is clear that \(A_i^c\) is the family of CFMSs containing \(A^c\)

\(\Rightarrow cl(A^c) = \{(\wedge (1 - \mu_{A_i}^j(x)), \wedge (1 - \mu_{A_i}^2(x)), \ldots, \wedge (1 - \mu_{A_i}^p(x))/x : x \in X\}\)

\(\Rightarrow cl(A^c) = \{(1 - \vee \mu_{A_i}^j(x)), (1 - \vee \mu_{A_i}^2(x)), \ldots, (1 - \vee \mu_{A_i}^p(x))/x : x \in X, i \in I\}\)

(2)

From (1) and (2)
\(cl(A^c) = (int(A))^c\).

4.5 Proposition: Let \((X,\Gamma)\) be FMT and \(A\) be an
FMS. Then \(int(A^c) = (cl(A))^c\).

Proof is similar to Proposition 4.4.

4.6 Proposition: Let \((X,\Gamma)\) be FMT and \(A\) and \(B\) are
OFMSs. Then the following properties hold.

1. \(int(A) \subseteq A\)
2. \(A \subseteq cl(A)\)
3. \(A \subseteq B \Rightarrow int(A) \subseteq int(B)\)
4. \(A \subseteq B \Rightarrow cl(A) \subseteq cl(B)\)
5. \(int(\emptyset) = \emptyset\)
6. \(int(\Gamma) = \Gamma\)

Proof follows from the definition.

4.7 Proposition: Let \((X,\Gamma)\) be FMT and \(A\) be an
FMS. Then \(A\) is CFMS if and only if \(A = cl(A)\).

Proof. Let \(A\) is CFMS. Then by proposition 4.6(2) we have
\(A \subseteq cl(A).\)
Since \(A\) is CFMS,
\(cl(A) \subseteq A.\)
From (3) and (4) \(cl(A) = A\)
Converse follows from the definition.

5. Continuous functions

In this section, we define fuzzy multiset
continuous function and equivalence relation of
fuzzy multiset continuous function.

5.1 Definition: Let \((X,\Gamma)\) and \((X,\Phi)\) be two FMTs.
A function \(f : X \rightarrow Y\) is said to be continuous if and
only if inverse image of each OFMS in \(\Phi\) is an
OFMS in \(\Gamma\).
5.1 Theorem: Let \((X,\Gamma)\) and \((X,\Phi)\) be two FMTs. A function \(f:X\to Y\) is said to be continuous if and only if inverse image of each CFMS in \(\Phi\) is a CFMS in \(\Gamma\).

Proof: Let \(f\) be continuous function and \(C\) be CFMS in \(\Phi\).

To prove \(f^{-1}(C)\) is closed, it is enough to show that \((f^{-1}(C))^c\) is open.

Now the inverse image of FMS \(C\) in \(Y\) under the mapping \(f\) is denoted by \(f^{-1}(C)\) where \(CM_{f^{-1}(C)}(x) = CM_{C}(f(x)) \).

\[ \Rightarrow f^{-1}(C) = \{ (CM_{C}(f(x)))/x : x \in X \} \]
\[ \Rightarrow (f^{-1}(C))^c = \{ 1 - CM_{C}(f(x))/x : x \in X \} . \]
(5)

Now \(f^{-1}(C^c) = \{ (CM_{C}(f(x)))/x : x \in X \} \)
\[ \Rightarrow f^{-1}(C^c) = \{ (1 - CM_{C}(f(x)))/x : x \in X \} \]
(6)

Where \(y = f(x)\)

From (5) and (6)
\[ f^{-1}(C^c) = (f^{-1}(C))^c. \]

Since \(f\) is continuous and \(C^c\) is open by definition of continuous function \(f^{-1}(C^c)\) is open and hence \((f^{-1}(C))^c\) is open.

Now assume that the inverse image of each CFMS in \(\Phi\) is a CFMS in \(\Gamma\).

To prove that \(f\) is continuous, it is enough to prove \(f^{-1}(O)\) is open for every OFMS in \(\Phi\).

\[ \Rightarrow O\text{ is an OFMS} \Rightarrow O^c\text{ CFMS} \]
\[ \Rightarrow f^{-1}(O^c)\text{ is CFMS by assumption. Since (7) is true for any CFMS,} (f^{-1}(O))^c\text{ is CFMS and hence} f^{-1}(O) \text{ is OFMS. Hence} f\text{ is continuous.} \]

5.2 Theorem: Let \((X,\Gamma)\) and \((X,\Phi)\) be two FMTs. A function \(f:X\to Y\) is said to be continuous if and only if each FMS \(A\) in \(X\), \(f(cl(A))\subseteq cl(f(A))\).

Proof: Let us assume that \(f\) is continuous.

For any FMS \(A\) in \(X\), \(f(cl(A))\subseteq cl(f(A))\) by proposition 4.8.2

\[ \Rightarrow A = f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A))) \]

Since \(f\) is continuous and \(cl(A)\) is closed, \(f^{-1}(cl(f(A)))\) is closed.

\[ \Rightarrow cl(A) \subseteq f^{-1}(cl(f(A))) \]

Since \(cl(A)\) is the smallest CFMS contains \(A\). Conversely assume that the given condition is true.

To prove \(f\) is continuous, Let \(C\) be a CFMS in \(\Phi\).

Then by assumption, \(f cl(f^{-1}(C))) \subseteq cl(f^{-1}(C))\)

Thus \(cl(f^{-1}(C)) \subseteq f^{-1}(C)\)

By the proposition 22(2), \(f^{-1}(C) \subseteq cl(f^{-1}(C))\)
\[ \Rightarrow f^{-1}(C) = cl(f^{-1}(C)) . \]
\[ \Rightarrow f^{-1}(C) \text{ is CFMS} \]

5.3 Theorem: Let \((X,\Gamma)\) and \((X,\Phi)\) be two FMTs. A function \(f:X\to Y\) is said to be continuous if and only if each FMS \(B\) in \(Y\), \(cl(f^{-1}(B))\subseteq cl^{-1}(cl(B))\).

Proof: Let us assume that \(f\) is continuous.

For any FMS \(A\) in \(X\), \(int(f(A))\subseteq f(int(A))\) by proposition 4.8.3
\[ \Rightarrow f^{-1}(int(f(A))) \subseteq int(f^{-1}(O)) \]

Since \(f\) is continuous and \(int(f(A))\) is OFMS, \(f^{-1}(int(f(A)))\) is OFMS.

But \(int(A)\) is the largest OFMS containing \(A\).
\[ \Rightarrow f^{-1}(int(f(A))) \subseteq int(A). \]
\[ \Rightarrow int(A) \subseteq f(int(A)). \]
Conversely assume that the given condition is true.

To prove \(f\) is continuous, let \(O\) be an OFMS in \(\Phi\).

Then by assumption, \(int(f^{-1}(O)) \subseteq f(int(f^{-1}(O)))\)
\[ \Rightarrow int(O) = f(int(f^{-1}(O))). \]
\[ \Rightarrow O \subseteq f(int(f^{-1}(O))). \]
Again by proposition 4.8.1 \(int(f^{-1}(O)) \subseteq f^{-1}(O)\)
\[ \Rightarrow int(f^{-1}(O)) = f^{-1}(O). \]
\[ \Rightarrow f^{-1}(O) \text{ is OFMS} \]
\[ \Rightarrow f\text{ is continuous.} \]

5.4 Theorem: Let \((X,\Gamma)\) and \((X,\Phi)\) be two FMTs. A function \(f:X\to Y\) is said to be continuous if and only if each FMS \(A\) in \(X\), \(f^{-1}(int(B))\subseteq int(f^{-1}(B))\).

Proof follows from theorem 5.2.

5.4 Theorem: Let \((X,\Gamma)\) and \((X,\Phi)\) be two FMTs. A function \(f:X\to Y\) is said to be continuous if and only if each FMS \(A\) in \(X\), \(f^{-1}(int(B))\subseteq int(f^{-1}(B))\).

Proof follows from theorem 5.2.

6 Fuzzy Multiset Subspace Topology

In this section, we give the definition of subspace of fuzzy multiset topological space and examples.

6.1 Definition: Let \((X,\Gamma)\) and \((Y,\Phi)\) be two FMTs. The topological space \(Y\) is called subspace of the topological space \(X\) if \(Y \subseteq X\) and \(\Phi \) precisely the subsets \(O\) of the form
\[ O' = O \cap Y \]
for some OFMS $O$ of $X$. Here we may say each
OFMS $O'$ of $Y$ is the restriction to $Y$ of an OFMS $O$
of $X$. $O'$ is also called relative OFMS in $Y$.

6.1 Example: Let $X=\{1, 2\}$ and defined the FMSs in
$X$ as follows,

For $n \in \mathbb{N}$

$F_n = \{(1, 1/2, 1/3, ..., 1/(n - 1))/1, (1/2, 1/3, 1/4, ..., 1/n)/2\}$ Let $\Gamma_1 = \{\overline{0}, \overline{1}\} \cup F_n$.

Then $(X, \Gamma_1)$ forms a subspace FMT of example
4.1.

6.1 Theorem: Let $(X, \Gamma)$ be FMT and let $Y$ be a
subset of $Y$. Define the collection $\Phi$ of subsets of $Y$
is the collection of subsets $O'$ of $Y$ of the form

$O'=O \cap Y$

where $O$ is an OFMS in $(X, \Gamma)$, Then $(Y \cup \{1\}, \Phi)$ is an
FMT and a subspace of $(X, \Gamma)$.

Proof: We have $\overline{0} = \overline{0} \cap Y$

Let $O_1', O_2', O_3', ..., O_n' \in \Phi$

Where $O_i' = O_i \cap Y$ for some $O_i' \in \Gamma$

Then $O_1' \cap O_2' \cap ... \cap O_n' = (O_1 \cap O_2 \cap ... \cap O_n) \cap Y$

Finally, Suppose that for each $\alpha \in I$, $O_\alpha' \in \Phi$

Thus for each $\alpha \in I$, $O_\alpha' = O_\alpha \cap Y$ for some $O_\alpha \in \Gamma$

Then $\cup O_\alpha' = \cup (O_\alpha \cap Y)$ $\subseteq \cup O_\alpha \cap Y \in \Phi$ Since

$\cup O_\alpha \subseteq \Gamma$

Hence $(Y \cup \{1\}, \Phi)$ is FMT and therefore a subspace of
$(X, \Gamma)$.

7 Conclusion

In this work we studied the topological
structure of Fuzzy Multisets. We introduced the
concept of fuzzy multiset topological space, open
fuzzy multisets, closed fuzzy multisets, interior of
fuzzy multiset, closure of fuzzy multisets and
discussed various properties. We discussed
continuous functions and various properties in term
of Fuzzy multisets.

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